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Path of light in near crack tip region in anisotropic medium and under mixed-mode loading

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Abstract

The theory governing the propagation of light in the region near a plane strain crack tip is extended to optically anisotropic media. As for the case of isotropic media, an incident wavefront of light is split into two independent wavefronts. The planes of polarization of the two wavefronts lie in planes parallel and perpendicular to the crack front only for specific crystal orientations relative to the crack front. The general functional forms for the index of refraction of both the parallel and perpendicular wavefronts are given in terms of the principal indices of refraction, orientation of the index ellipsoid, and the direction of propagation of the wavefront. It is demonstrated how the principal indices of refraction and the orientation of the index ellipsoid can be calculated from the results of a finite element analysis and the coefficients of the elasto-optic tensor for a specific case. The characteristic equations which govern the propagation of light in anisotropic media are solved for a specific case to calculate the path of light relative to a crack tip in a bicrystal specimen. We also report the path of light rays approaching a mixed Mode I and Mode II crack tip in a material that is optically isotropic in its unstrained state.

Keywords: Anisotropic; Crack tip; Crystals; Experimental techniques; Elasto-optic tensor

1. Introduction

Crack opening interferometry (e.g. Liechti, 1993; Kysar, 2000a) is an optical technique commonly used to measure normal crack opening displacement profiles of a crack in transparent materials. In principle, a wavefront of light that approaches the two crack flanks at approximately normal incidence partially reflects off both crack flanks. The reflected wavefronts subsequently interfere to form a set of fringes that are loci of constant normal crack opening displacement which can be interpreted in terms of normal crack opening displacement profile. Another optical technique used to measure crack tip parameters in transparent materials is the method of caustics (e.g. Kalthoff, 1993; Rosakis, 1993) in which incident light is transmitted parallel to the crack flanks and crack front. Proper interpretation of this technique requires that the change

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in optical path length of the light due to stress-induced changes of index of refraction be considered in the region of singular stress at the crack tip.

Fowlkes (1975) and Liang and Liechti (1995) raised the issue that stress induced changes in index of refraction at the tip of a crack might also affect the correct interpretation of fringes obtained from crack opening interferometry. They postulated that the inhomogeneous index of refraction could cause a noticeable "mirage" effect in which the light approaching the tip bends or deflects significantly from a straight line path. This phenomenon could potentially invalidate the assumption of normal crack flank incidence implicit in crack opening interferometry. In addition it could cause the interference fringes to stretch or contract, which would lead to an incorrect interpretation of the crack opening profile.

Kysar (1998) addressed these questions by considering the properties of light as it approaches a plane strain crack tip in the plane of plane strain. The wavefront of light splits into two independent wavefronts: one wavefront polarized in a plane parallel to the crack front (i.e. electric displacement vector parallel to the crack front); the other polarized in a plane perpendicular to the crack front. The wavefronts are referred to, respectively, as the parallel wavefront and the perpendicular wavefront. The exact expression for index of refraction of both wavefronts as a function of position and direction of propagation was derived from the singular strain field for Mode I loading and the elasto-optic tensor. The index of refraction of the parallel wavefront is isotropic, while the index of refraction of the perpendicular wavefront is anisotropic. In addition, since the light ray associated with each wavefront follows a separate path, each deflects a different amount from a nominally straight line path. Kysar (1998) concluded that, depending upon the material, the apparent position of the crack tip might change by several microns due to the mirage effect. However, the assumption of normal crack flank incidence is not violated, and stretching and contraction of the fringes is expected to be negligible. Therefore the stress-induced changes in index of refraction is not expected to affect the interpretation of the interference fringes in terms of crack opening profile.

The analysis by Kysar (1998) assumed the transparent material to be optically isotropic in its unstrained state (i.e. a polymer or a glass). In the present paper, the analysis is extended to consider the propagation of light at normal incidence to the crack flanks in materials that are optically anisotropic in the unstrained state (i.e. single crystals). It will be shown that an incident wavefront of light, again, splits into two independent wavefronts, but that the two planes of polarization do not necessarily lie parallel and perpendicular to the crack front. However in special cases the planes of polarization can be made to lie parallel and perpendicular to the crack front and in this configuration, it is straightforward to write a general expression for the index of refraction experienced by both wavefronts. The characteristic set of equations of geometrical optics in anisotropic media can then be solved numerically to determine the paths of the two light rays. To illustrate the concepts, the paths of light rays approaching an interfacial crack tip in a copper/sapphire bicrystal specimen used in a study by Kysar (2000b) are calculated.

In addition, Escobar (1995) attempted to measure crack parameters with a new optical technique that is similar in spirit to both crack opening interferometry and the method of caustics. As with crack opening interferometry, the light approaches the crack flanks with normal incidence. But instead of striking the crack flanks, the light passes directly in front of the crack tip and, in doing so, changes slightly its angle of incidence due to stress-induced changes of index of refraction. She attempted to relate the angle at which the light exited the material to the applied stress intensity factor. To facilitate calculation of the change of angle, we extend in the present study the Kysar (1998) analysis to the case of a mixed Mode I and Mode II crack in materials that are optically isotropic in their unstrained state.

The format of the present paper is as follows. Section 2 briefly reviews the behavior of light in an optically anisotropic medium and shows under what conditions the planes of polarization of an incident wavefront of light will be parallel and perpendicular to the crack front both before and during loading. Section 3 presents the general functional form of the index of refraction for such a case and discusses the characteristic set of equations which determines the path of a light ray in an anisotropic, inhomogeneous medium. In Section 4 the concepts are illustrated by calculating the path of the light approaching the flanks of an interfacial crack in a copper/sapphire bicrystal. In Section 5, we generalize the results of light approaching a crack flank in an optically isotropic material to the case of a mixed Mode I and Mode II loading. Section 6 discusses the results and presents conclusions.

2. Planes of polarization in near crack tip region of optically anisotropic medium

It is well established (e.g. Born and Wolf, 1993) that a wavefront of light in a transparent, non-magnetic, homogeneous, anisotropic medium splits into two independent wavefronts which are linearly polarized in orthogonal planes. A geometrical construction called the index ellipsoid is commonly used to determine the planes of polarization and the indices of refraction of the two wavefronts using the known direction of wavefront propagation and the directions and values of the principal indices of refraction. The principal indices of refraction of the index ellipsoid are defined as $n_k^2 = 1/B_k$, $k = 1, 2, 3$, where B_k are the principal values of the second order relative dielectric impermeability tensor B_{ij}; the directions of n_k correspond to those of B_k. The two planes of polarization and the indices of refraction associated with a given direction of light wave propagation are determined (e.g. Nye, 1985) by forming the elliptical cross-section of the index ellipsoid that is perpendicular to the wavefront normal and passes through the center of the index ellipsoid. The lengths of the semi-axes of the elliptical cross-section are equal to the indices of refraction, and the directions indicate planes of polarization. In an elastic medium, the change in B_{ij} is a function of the applied strain, ε_{ij}, via the relationship $\Delta B_{ij} = p_{ijkl}\varepsilon_{kl}$, where p_{ijkl} is the fourth order elasto-optic tensor (e.g. Nye, 1985). Thus a strain can change the shape and orientation of the index ellipsoid thereby altering the planes of polarization and indices of refraction of an incident wavefront. These concepts are generalizations of the Maxwell–Neumann stress-optic law (e.g. Kim et al., 1987) for artificial birefringence.

In crack opening interferometry it is desirable to choose the crystal and its orientation relative to the crack front so as to obtain planes of polarization that are parallel and perpendicular to the crack front, because in this case the indices of refraction can be expressed in simple functional form. Two conditions must be satisfied for this to occur in a plane strain crack. The first necessary condition is that one principal axis of B_{ij} in the unstrained state must be parallel to the crack front. This ensures that the planes of polarization lie parallel and perpendicular to the crack front both in the unstrained state as well as in the rotated state induced by the anti-symmetric part of the displacement gradient. In addition, since an applied strain acts through p_{ijkl} to change the components of B_{ij}, the form of p_{ijkl} determines whether the planes of polarization change or remain the same under loading. Under plane strain conditions, the only non-zero strains are the in-plane shear and normal strains. Thus, the second necessary condition requires that these in-plane strains do not change an off-diagonal, out-of-plane component of B_{ij}. If both conditions are satisfied, the planes of polarization will lie parallel and perpendicular to the crack front both before and during loading. This does not say, however, that one cannot calculate the paths of the light rays for any other crystal orientation. But, since the planes of polarization would change with load level, the expressions for the indices of refraction would become cumbersome.

3. Indices of refraction and geometrical optics in anisotropic media

We assume that a plane strain crack exists in a transparent medium in the x_1–x_2 plane; the crack front coincides with the x_3-axis and that the planes of polarization of the incident wavefronts of light are parallel and perpendicular to the crack front. Fig. 1 shows the cross-section of the index ellipsoid in the x_1–x_2 plane. The orientation of the index ellipsoid relative to the x_1-axis is denoted by ϕ and the angle ω specifies the direction of the wavefront normal of light propagating in the x_1–x_2 plane. The position of the wavefront is

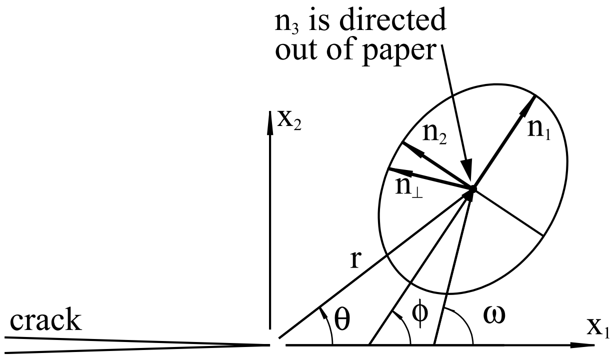

Fig. 1. Orientation of index ellipsoid relative to plane strain crack, with ω equal to direction of wavefront normal, ϕ gives orientation of index ellipsoid, r and θ indicate position of wavefront, and n_1, n_2 and n_3 are principal indices of refraction.

specified by polar position coordinates r and θ. The principal indices of refraction n_1 and n_2 lie in the plane; n_3 is directed out of the plane.

Kysar (1998) showed that in this instance, the index of refraction of the parallel wavefront, denoted as $n_{||}$, is a function only of position. However the index of refraction of the perpendicular wavefront, denoted as $n_{\perp}$, is a function of both position and direction of the wavefront normal. The functional forms of the indices of refraction are

$$\begin{aligned} n_{||} &= n_3, \\ n_{\perp}^2 &= \tfrac{1}{2}(n_1^2 + n_2^2) - \tfrac{1}{2}(n_1^2 - n_2^2)\cos[2(\omega - \phi)]. \end{aligned} \tag{1}$$

The path of the light ray associated with each wavefront is found by solving the characteristic set of equations of geometrical optics in anisotropic media (e.g. Kravtsov and Orlov, 1990 and references therein). When these equations are applied to in-plane propagation of light in an optically transparent medium with index ellipsoid as shown in Fig. 1, it can be shown (Kysar, 1998) that the characteristic set of equations is

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}s} &= \cos\alpha\left[\frac{p_1}{n} + \frac{p_2}{n}\left(\frac{1}{n}\frac{\partial n}{\partial\omega}\right)\right], \\ \frac{\mathrm{d}x_2}{\mathrm{d}s} &= \cos\alpha\left[\frac{p_2}{n} - \frac{p_1}{n}\left(\frac{1}{n}\frac{\partial n}{\partial\omega}\right)\right], \end{aligned} \tag{2}$$

$$\begin{aligned} \frac{\mathrm{d}p_1}{\mathrm{d}s} &= \cos\alpha\left[\frac{\partial n}{\partial x_1}\right], \\ \frac{\mathrm{d}p_2}{\mathrm{d}s} &= \cos\alpha\left[\frac{\partial n}{\partial x_2}\right], \end{aligned} \tag{3}$$

$$\cos\alpha = \left[1 + \left(\frac{1}{n}\frac{\partial n}{\partial\omega}\right)^2\right]^{-(1/2)}, \tag{4}$$

where the subscript on the index of refraction, n, has been dropped and $\omega = \mathrm{atan}(p_2/p_1)$.

Physically, the variables in the characteristic equations can be interpreted as follows: x_i corresponds parametrically to the path of the light ray in the x_1–x_2 plane with path length s, $p_i = nm_i$ is the "slowness" vector in the direction of the wave normal, m_i is the unit vector normal to wavefront, and α is the angle between wavefront normal and light ray, which do not coincide for light propagation in an anisotropic medium. Since there are two independent wavefronts, the characteristic set of equations must be solved twice: once for the parallel wavefront and once for the perpendicular wavefront. We should note that there is potential for confusion given that it is customary to write the elasto-optic tensor as p_{ijkl} and the slowness vector as p_i; the meaning should be clear from context.

4. Path of light rays in a copper/sapphire bicrystal

The specific bicrystal configuration considered in this study is shown in Fig. 2. A single copper crystal, 3 mm in height, is diffusion bonded to a single crystal sapphire that is 1 mm in height. A crack is introduced at the interface and the system is loaded with a 125 $\mathrm{N\,mm^{-1}}$ moment such that the upper surface of the copper crystal is in compression. Ahead of the crack, the moment is applied over the entire cross-section; behind the crack, the load is applied only to the copper crystal. It is desired to measure experimentally the normal crack opening displacement profile with interferometry by shining a light, as shown in Fig. 2, through the transparent sapphire. The light partially reflects off each crack flank and when recombined, forms a set of interference fringes that can be interpreted in terms of normal crack opening displacement. In the case considered here, the incident wavefront initially has normal incidence (i.e. parallel to the x_2-axis). Our goal in the present study is to calculate the path of the light as it approaches the crack tip to see if the stress-induced changes of index of refraction affect the necessary interpretation of interference fringes.

A finite element analysis of this configuration in plane strain was performed in a previous study (Mesarovic and Kysar, 1996) using ABAQUS/Standard (1993) in order to determine the strain state. The smallest element size in the near tip region was a square of side 0.2 μm. Two different cases are considered. In one, the copper is allowed to deform plastically; in the other, the copper remains elastic. To model plastic flow in the copper crystal, the analysis used a subroutine (Huang, 1991) which accounts for the single crystal plasticity; the parameters of the hardening model are discussed in Mesarovic and Kysar (1996). In both cases, the sapphire was assumed to remain elastic. The results of these analyses are used to investigate the behavior of the light close to the crack tip.

Sapphire (α-Al_2O_3) is a trigonal crystal of point group $\bar{3}$ m and hence its index ellipsoid is an ellipsoid of revolution about the optical axis which coincides with the three-fold symmetry axis. The (0 0 0 1) basal plane, which is perpendicular to the optical axis, is bonded to the $(\bar{2}\,2\,\bar{1})$ plane of the copper crystal; the sapphire $[1\,1\,\bar{2}\,0]$ is parallel to the crack front. The unstrained index ellipsoid relative to the coordinate

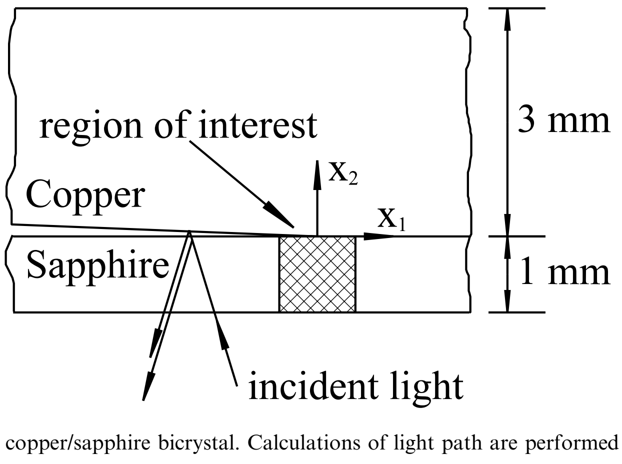

Fig. 2. Interface crack in copper/sapphire bicrystal. Calculations of light path are performed in cross-hatched region.

system of Fig. 2 is specified by the principal indices of refraction $n_1 = n_3 = 1.758$ and $n_2 = 1.766$ (Akulyenok et. al., 1970). Appendix A shows that the sapphire in this orientation satisfies the requirements of Section 2 so that the incident light is polarized parallel and perpendicular to the crack front.

It is necessary to determine the principal indices of refraction n_1, n_2, and n_3, and as well as the orientation of the index ellipsoid ϕ, in the near crack tip region of the copper/sapphire specimen indicated in Fig. 2. This is accomplished by calculating the changes in B_{ij} using experimentally determined values of the elasto-optic coefficients (reported in Appendix A) and strains from the finite element analyses. Next the eigenvalues and eigenvectors of the "strained" B_{ij} are calculated. The eigenvalues correspond to the principal indices of refraction and the angle ϕ is the angle of rotation in the x_1–x_2 plane between the eigenvectors of the "unstrained" index ellipsoid and the "strained" index ellipsoid. In addition, local crystalline lattice rotation (i.e. the non-symmetric part of the displacement gradient) contributes to ϕ. However the FEM results show that the lattice rotations very near the crack tip are on the order of $\phi = 0.05°$, whereas the rotation of the index ellipsoid due to the strain field is on the order of $\phi = 4°$. Therefore we neglect the rotation of the index ellipsoid due to local lattice rotations in what follows. Hence the indices of refraction, $n_{\parallel}$ and $n_{\perp}$, and their derivatives are calculated from Eq. (1) using the numerically obtained n_1, n_2, n_3, and ϕ as well as the orientation of the wavefront normal, ω.

We are now prepared to integrate the characteristic set of Eqs. (2)–(4) to obtain the paths of the *light rays* in the copper/sapphire specimen. Following Fowlkes (1975), Fig. 3 shows the geometry considered. Initially the unit normal wavefront vector, m_i, is directed in the positive x_2-direction. We introduce a new variable, ξ, which corresponds to the deflection of light from the nominal straight line path and write the x_1-component of the position vector of the light ray path as $x_1 = -(\xi(x_2) + x_1^0)$. A positive value of ξ indicates that the light deflects in the negative x_1-direction. The initial position, x_1^0, is arbitrary, but we will consider only the special case of $x_1^0 = 0$. Thus the deflection of the light ray, rather than its absolute position, is calculated which is advantageous because the magnitude of the light deflection, ξ, is expected to be very small as compared to the path length, s.

The fourth-order Runge–Kutta technique is used to integrate the set of ordinary differential equations posed by Eqs. (2)–(4) subject to the initial conditions $x_1^0 = 0$ mm, $x_2^0 = -1$ mm, $\xi(x_2^0) = 0$, and $p_2^0 = n(x_1^0, x_2^0, \omega = \pi/2)$ which implies that $p_1^0(x_1^0, x_2^0, \omega = \pi/2) = 0$. The predicted paths of the light rays associated with both the perpendicular and parallel wavefronts as they traverse the sapphire are shown in Fig. 4. The negative of the path deviation, $-\xi(x_2)$, is plotted on the abscissa and the position along the x_2-axis within the sapphire is plotted on the ordinate for both the elastic and plastic FEM calculations. The figure

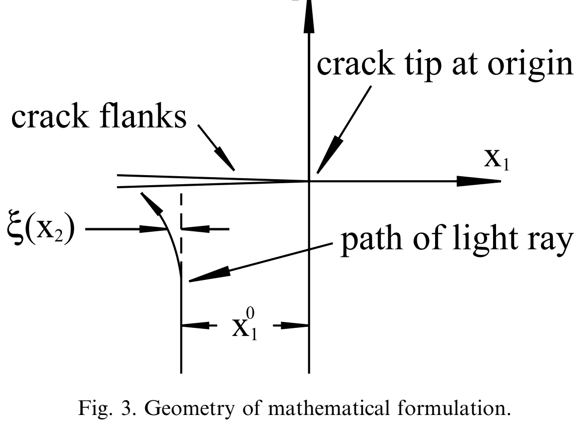

Fig. 3. Geometry of mathematical formulation.

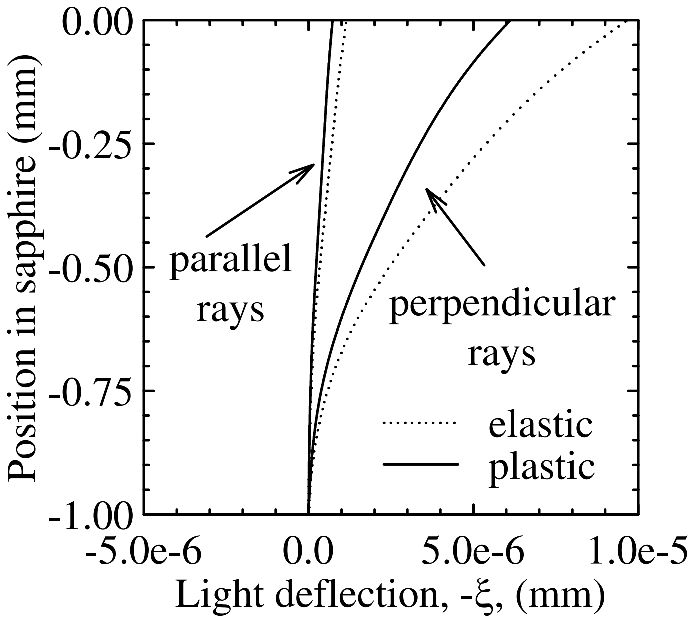

Fig. 4. Paths of light parallel and perpendicular light rays in copper/sapphire bicrystal assuming both plastic and elastic copper behavior.

can be interpreted as being an exaggerated view of the predicted light path as the light enters the sapphire at $x_2^0 = -1$ mm and propagates to the crack flanks at $x_2^0 = 0$ mm. The final deviation of both light rays is on the order of $O(10^{-6})$ mm. The maximum angular deviation of a light ray from the optical axis is on the order of $O(10^{-4})$ rad. Therefore light deflection and angular deviation are sufficiently small so as not to affect the interpretation of the interference fringes.

5. Indices of refraction under mixed Mode I and Mode II loading

We now turn our attention to the case of a mixed Mode I and Mode II plane strain crack that exists in a medium which has mechanical and optical isotropy and homogeneity in the unstrained state. Further, the medium exhibits both linear elastic and linear optic behavior. We report the expressions for the indices of refraction of both the parallel and perpendicular wavefronts and calculate the paths of the associated light rays as they approach the crack tip. Following Kysar (1998), from the known mixed mode elastic strain field it can be shown for plane strain that, within the near crack tip region where the singular strain field dominates, the indices of refraction for the perpendicular and parallel wavefronts are

$$n_{\|} = n_0\left\{1 - \sqrt{2}c_{\|}r^{-(1/2)}\left[q_{\mathrm{I}}\cos\left(\frac{\theta}{2}\right) - q_{\mathrm{II}}\sin\left(\frac{\theta}{2}\right)\right]\right\}, \tag{5}$$

$$n_{\perp} = n_0\left\{1 - \sqrt{2}c_{\perp}r^{-(1/2)}\left[q_{\mathrm{I}}\cos\left(\frac{\theta}{2}\right) - q_{\mathrm{II}}\sin\left(\frac{\theta}{2}\right)\right] - \sqrt{2}c_{\perp}r^{-(1/2)}\beta\sqrt{q_{\mathrm{II}}^2\sin^2\theta + (q_{\mathrm{I}}\sin\theta + 2q_{\mathrm{II}}\cos\theta)^2}\cos\left[2(\omega - \phi)\right]\right\}, \tag{6}$$

where

$$\Phi = \frac{1}{2}\operatorname{atan}\left\{\frac{\sin\theta\cos(3\theta/2) + 2\tan\psi\cos(\theta/2)[1 - \sin(\theta/2)\sin(3\theta/2)]}{-\sin\theta\sin(3\theta/2) - 2\tan\psi\sin(\theta/2)[1 + \cos(\theta/2)\sin(\theta/2)}\right\}$$

and $n_{\|}$, $n_{\perp}$ are indices of refraction, respectively, of parallel and perpendicular wavefronts, r, θ is polar position coordinates, $K_{\mathrm{I,II}}$ are Mode I and Mode II stress intensity factors, $q_{\mathrm{I,II}} = (1-2v)(1+v)K_{\mathrm{I,II}}/E\sqrt{\pi}$, Mode I and Mode II strain intensity factor, $\psi = \operatorname{atan}\left(\frac{K_{\mathrm{II}}}{K_{\mathrm{I}}}\right)$, loading phase angle, E is Young's modulus, v is Poisson's ratio, $c_{\|} = \frac{1}{2}p_{12}n_0^2$, $c_{\perp} = \frac{1}{4}(p_{11}+p_{12})n_0^2$, $\beta = (P_{11}-p_{12})/2(1-2v)(p_{11}+p_{12})$, strain induced optical anisotropy, p_{11}, p_{12} are elasto-optical coefficients for optically isotropic material, n_0 is index of refraction in unstrained state, ω is angle that wavefront normal vector makes with line $\theta = 0$.

Representative values of these parameters are tabulated in Kysar (1998). We note that the dimensionless combination $c_{\perp}(q_{\mathrm{I}} + q_{\mathrm{II}})r^{-(1/2)} \ll 1$ for any radius r that is larger than an atomic spacing. Further the strain induced optical anisotropy parameter, β is of order 10^{-1}. We are now prepared to solve for the path of the light ray for both the perpendicular and parallel wavefronts. Since $n_{\perp}$ reduces to $n_{\|}$ if we set $\beta = 0$ and $c_{\perp} = c_{\|}$, it suffices to calculate the path of the light ray associated with $n_{\perp}$.

First we simplify the characteristic set of equations of geometrical optics in anisotropic media in Eqs. (2)–(4) to an intuitive form that holds for the case of "small anisotropy". For $\alpha \ll 1$, the expression for the angle between the wavefront normal and the direction of the associated light ray in Eq. (4) can be rewritten using the binomial theorem and the Taylor expansion for cosine as

$$\alpha \approx \frac{1}{n_{\perp}}\frac{\partial n_{\perp}}{\partial\omega}. \tag{7}$$

The characteristic system of differential Eqs. (2) and (3) can then be combined and rearranged to obtain, with $\cos\alpha \approx 1$

$$\frac{\mathrm{d}^2x_1}{\mathrm{d}s^2} + \left(\frac{1}{n_{\perp}}\frac{\partial n_{\perp}}{\partial s}\right)\frac{\mathrm{d}x_1}{\mathrm{d}s} = \frac{1}{n_{\perp}}\frac{\partial n_{\perp}}{\partial x_1} + \alpha\frac{1}{n_{\perp}}\frac{\partial n_{\perp}}{\partial x_2} + \frac{p_2}{n_{\perp}}\frac{\mathrm{d}\alpha}{\mathrm{d}s}, \tag{8}$$

$$\frac{\mathrm{d}^2x_2}{\mathrm{d}s^2} + \left(\frac{1}{n_{\perp}}\frac{\partial n_{\perp}}{\partial s}\right)\frac{\mathrm{d}x_2}{\mathrm{d}s} = \frac{1}{n_{\perp}}\frac{\partial n_{\perp}}{\partial x_2} - \alpha\frac{1}{n_{\perp}}\frac{\partial n_{\perp}}{\partial x_1} - \frac{p_1}{n_{\perp}}\frac{\mathrm{d}\alpha}{\mathrm{d}s}. \tag{9}$$

We consider a wavefront of light which propagates nominally along the line $\theta = -\pi/2$ in the direction $\omega = \pi/2$ relative to Fig. 1. Eq. (8) can be simplified by applying to it the geometry of Fig. 4 and substituting and $\xi = -x_1$ and $r = -s$, with $x_1^0 = 0$. Further it can be linearized by replacing $n_{\perp}$ with n_0 whenever $n_{\perp}$ occurs in a denominator and by assuming that $p_2 \approx n_{\perp}$ for $\alpha \ll 1$. Finally, if both Eqs. (8) and (9) are made non-dimensional by scaling r with r_0 and ξ with q_{I}^2, it can be shown that the second term on both sides of Eq. (8) are negligible when compared with the other terms. We can then write (in dimensional form) the governing equation in the limit of small anisotropy for the deflection of the light as it approaches a crack tip

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}r^2} = \frac{\mathrm{d}\alpha}{\mathrm{d}r} - \frac{1}{n_{\perp}}\frac{\partial n_{\perp}}{\partial x_1} \tag{10}$$

subject to the initial conditions

$$\left.\frac{\mathrm{d}\xi}{\mathrm{d}r}\right|_{r=r_0} = \alpha(r_0),$$

$$\xi(r = r_0) = 0,$$

where from Eq. (7)

$$\alpha(r) = -2c_{\perp}(q_{\rm I} + q_{\rm II})\beta r^{-(1/2)}.$$

The initial radius, r_0, is the largest radius at which the singular term in the strain field dominates (i.e. the radius of the K-field) and is a function of the geometry of a specimen. The initial condition for the slope is determined by the angle α at $r = r_0$. The governing equation, Eq. (10), has a remarkable simple form which concisely exposes the underlying physics of light propagation in an inhomogeneous medium with small anisotropy. The first term on the right-hand side is the contribution of the anisotropy to the light deflection. The second term is the contribution due to the inhomogeneity.

Now by substituting Eq. (6) into Eq. (10), the rate of deflection and the total deflection of the light ray for the perpendicular wavefront subject to the initial conditions is found by simple integration. The results (expressed in dimensional variables) valid at any radius $0 < r < r_0$ are

$$\frac{\mathrm{d}\xi_{\perp}}{\mathrm{d}r} = -2c_{\perp}\beta(q_{\rm I} + q_{\rm II})r^{-(1/2)} - c_{\perp}r_0^{-(1/2)}\left\{[q_{\rm I}(1-3\beta) - q_{\rm II}(1-\beta)]\left[\left(\frac{r}{r_0}\right)^{-(1/2)} - 1\right]\right\}, \quad (11)$$

$$\xi_{\perp}(r) = -4\beta c_{\perp}(q_{\rm I} + q_{\rm II})r_0^{1/2}\left[\left(\frac{r}{r_0}\right)^{1/2} - 1\right] + c_{\perp}r_0^{1/2}[q_{\rm I}(1-3\beta) + q_{\rm II}(1-\beta)]\left[\left(\frac{r}{r_0}\right)^{1/2} - 1\right]^2. \quad (12)$$

The equation governing the light path in the x_2-direction can be obtained in a similar manner from Eq. (9). It is automatically satisfied to the order $\mathrm{O}(c^2q^2r_0^{-1})$, so Eq. (10) suffices to calculate the light path. The analogous expressions for $\mathrm{d}\xi_{\|}/\mathrm{d}r$ and $\xi_{\|}$ are obtained from Eqs. (11) and (12) by setting $\beta = 0$ and substituting $c_{\|}$ for $c_{\perp}$. Kysar (1998) shows that the rate of deflection remains finite for all $r > \lambda$, where λ is the wavelength of the incident light.

As the light ray approaches the crack flanks (i.e. $r \to 0$), the total light deflection becomes

$$\xi_{\|}^{\rm total} = c_{\|}(q_{\rm I} - q_{\rm II})r_0^{1/2}, \quad (13)$$

$$\xi_{\perp}^{\rm total} = c_{\perp}[q_{\rm I}(1+\beta) - q_{\rm II}(1-5\beta)]r_0^{1/2} \quad (14)$$

which are valid for $q_{\rm I} > 0$ and for all $q_{\rm II}$. Care must be taken, though, when applying these equations when there is significant Mode II loading. Physically, such a loading often results in contact between the crack flanks which would change the stress field and render the equations inapplicable. Other than this point, the applicability of these solutions and the implications that they have on the field of crack opening interferometry are addressed in Kysar (1998).

6. Summary and conclusions

The theory governing the propagation of light in the region near a plane strain crack tip has been extended to optically anisotropic single crystals. As with the case of optically isotropic media, the incident wavefront of light splits into two independent wavefronts. For specific crystal orientations relative to the crack front, the planes of polarization of the two wavefronts are parallel and perpendicular to the crack front.

The general functional forms for the index of refraction of both the parallel and perpendicular wavefronts are given in terms of the principal indices of refraction, orientation of the index ellipsoid, and the direction of propagation of the wavefront. It is demonstrated how the principal indices of refraction and the orientation of the index ellipsoid can be calculated from results of a finite element analysis and the coefficients of the elasto-optic tensor for the specific case of light propagating through a sapphire crystal. The rotation of the index ellipsoid due to local lattice rotation of the sapphire crystal is shown to be negligible in this case.

The characteristic set of ordinary first order differential equations which governs the path of the light ray through the transparent material is presented. They are solved numerically to calculate the path of the light rays for a copper/sapphire bicrystal. The deflection of incident light close to the crack tip of the copper/sapphire bicrystal is two orders of magnitude smaller than can be detected by optical methods. The maximum angle of deflection of light from the straight line path is on the order of $O(10^{-4})$ rad. The effect will neither shift the apparent crack tip position nor will it change the interpretation of the normal crack opening displacement.

We also report the indices of refraction and the path of light rays as they approach a crack tip in a medium that is optically isotropic in its unstrained state.

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Appendix A. Planes of polarization of light in copper/sapphire specimen

This appendix illustrates how to apply the criteria of Section 2 to determine if the two planes of polarization of the wavefronts of light lie in planes that are perpendicular and parallel to a plane strain crack front. The first condition states that one of the principal axes of B_{ij} must be parallel to the crack front. The second condition requires that the in-plane strains do not change an off-diagonal, out-of-plane component of B_{ij}. An example calculation using a copper/sapphire bicrystal system is shown to illustrate the procedure.

It is convenient to express the elasto-optic tensor in terms of its matrix representation. Sapphire (α-Al_2O_3) is a trigonal crystal which implies that it is optically uniaxial. If, as is conventional, the optical axis is chosen to coincide with the crystallographic x_3-axis, ΔB_{ij} as a function of the strain state ε_{ij} for the $\overline{3}$ m point group is

$$\begin{bmatrix} \Delta B_1 \\ \Delta B_2 \\ \Delta B_3 \\ \Delta B_4 \\ \Delta B_5 \\ \Delta B_6 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & 0 & 0 \\ p_{12} & p_{11} & p_{13} & -p_{14} & 0 & 0 \\ p_{31} & p_{31} & p_{33} & 0 & 0 & 0 \\ p_{41} & -p_{41} & 0 & p_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{44} & p_{41} \\ 0 & 0 & 0 & 0 & p_{14} & p \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} \tag{A.1}$$

where $p = (p_{\mathrm{II}} - p_{12})/2$. Nye (1985) discusses how to convert from the tensor to the matrix notation.

If the x_3-axis of the sapphire is chosen to be perpendicular to the plane strain crack front with the x_1-axis parallel to the crack front, the first condition is satisfied. In that case the strain vector is $\{0, \varepsilon_2, \varepsilon_3, \varepsilon_4, 0, 0\}^{\mathrm{T}}$, which induces $\{\Delta B_1, \Delta B_2, \Delta B_3, \Delta B_4, 0, 0\}^{\mathrm{T}}$. Since the only off-diagonal term B_{ij} that changes is the component which corresponds to in-plane rotations of the eigenvectors, this orientation satisfies the second condition.

Kaminskii (1981) reports the experimentally determined values of the dimensionless elasto-optic tensor for sapphire as $p_{11} = -0.25$, $p_{33} = -0.23$, $p_{12} = -0.038$, $p_{13} = 0.005$, $p_{31} = -0.032$, $p_{44} = -0.1$, $p_{14} = 0.02$, and $p_{41} = 0.01$.

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